

Reading Debrief

- Discuss Activities 10.5.2 & 10.5.3 w/ your group.
- Questions about the chain rule?
- Class discussion about Preview Activity 10.6.1.

b. Suppose that $z = x^2 - 2xy^2$ and that

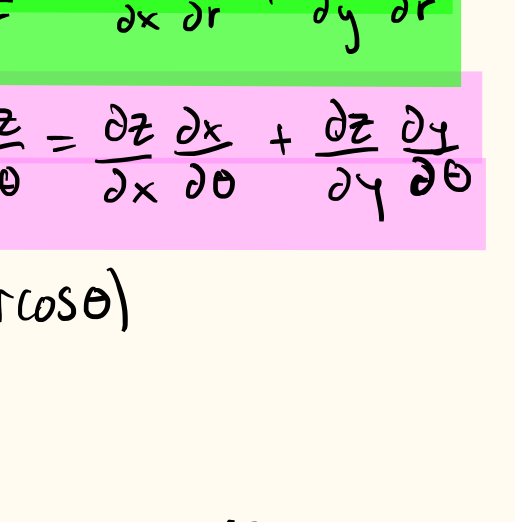
$$x = r \cos(\theta)$$

$$y = r \sin(\theta).$$

i. Construct a tree diagram representing the dependencies of z on x and y and x and y on r and θ .

ii. Use the tree diagram to find $\frac{\partial z}{\partial r}$.

iii. Now suppose that $r = 3$ and $\theta = \pi/6$. Find the values of x and y that correspond to these given values of r and θ , and then use the Chain Rule to find the value of the partial derivative $\frac{\partial z}{\partial r}(3, \pi/6)$.



$$\begin{aligned} \text{iii)} \quad \frac{\partial z}{\partial x} &= 2x - 2y^2 & \frac{\partial x}{\partial r} &= -r \sin \theta \\ \frac{\partial z}{\partial y} &= -4xy & \frac{\partial y}{\partial r} &= r \cos \theta \\ x(3, \pi/6) &= 3 \cdot \frac{\sqrt{3}}{2} & y(3, \pi/6) &= \frac{3}{2} \end{aligned}$$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r}$$

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$$\frac{\partial z}{\partial r} = (2x - 2y^2)(-r \sin \theta) + (-4xy)(r \cos \theta)$$

Preview Activity 10.6.1. Let's consider the function f defined by

$$f(x, y) = 30 - x^2 - \frac{1}{2}y^2.$$

c. Now, rather than walking due east or due north, let's suppose that the person is walking with velocity given by the vector $\mathbf{v} = \langle 3, 4 \rangle$, where time is measured in seconds. Note that the person's speed is thus $|\mathbf{v}| = 5$ feet per second. Find parametric equations for the person's path; that is, parametrize the line through $(2, 1)$ using the direction vector $\mathbf{v} = \langle 3, 4 \rangle$. Let $x(t)$ denote the x -coordinate of the line, and $y(t)$ its y -coordinate. Make sure your parameterization places the walker at the point $(2, 1)$ when $t = 0$.

d. With the parameterization in (c), we can now view the temperature f as not only a function of x and y but also of time, t . Hence, use the chain rule to determine the value of $\frac{df}{dt}(t_0)$. What are the units on your answer? What is the practical meaning of this result?

$$\mathbf{r}(t) = \langle 2, 1 \rangle + \langle 3, 4 \rangle t = \langle 2 + 3t, 1 + 4t \rangle$$

parametrizes the person's path.

(d) Viewing temp as a function of t means we consider the composite function $f(x(t), y(t))$. To compute $\frac{df}{dt}$ we use the chain rule:

$$\begin{aligned} \frac{df}{dt} &= \left(\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \right) \Big|_{t=0} \\ &= (-2x \cdot 3 - y \cdot 4) \Big|_{t=0} \\ &= -2(2) \cdot 3 - 1 \cdot 4 = -16 \text{ } ^\circ\text{C/s} \end{aligned}$$

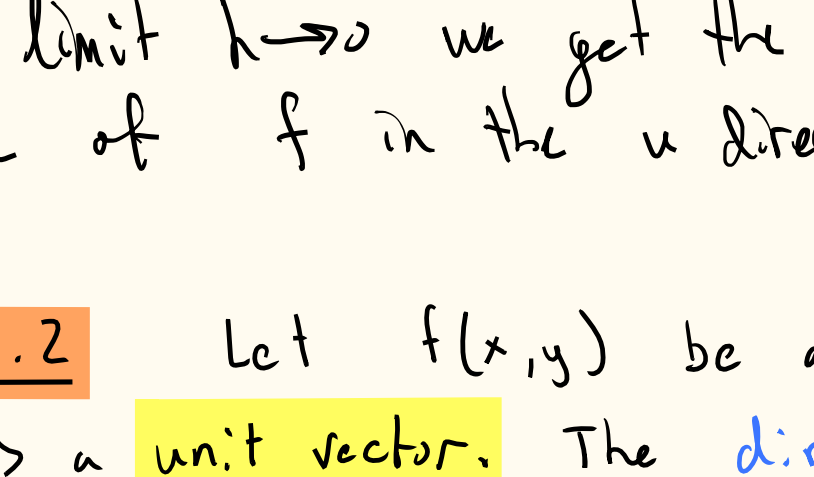
The practical meaning is that when the person starts their walk at $t=0$, the instantaneous rate of change of temp w/ respect to time is -16 $^\circ\text{C/s}$.

Section 10.6.1

Directional Derivative

We want measure the rate of change of a function $f(x, y)$ as we travel in a specific direction given by a unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ at a point (x_0, y_0) .

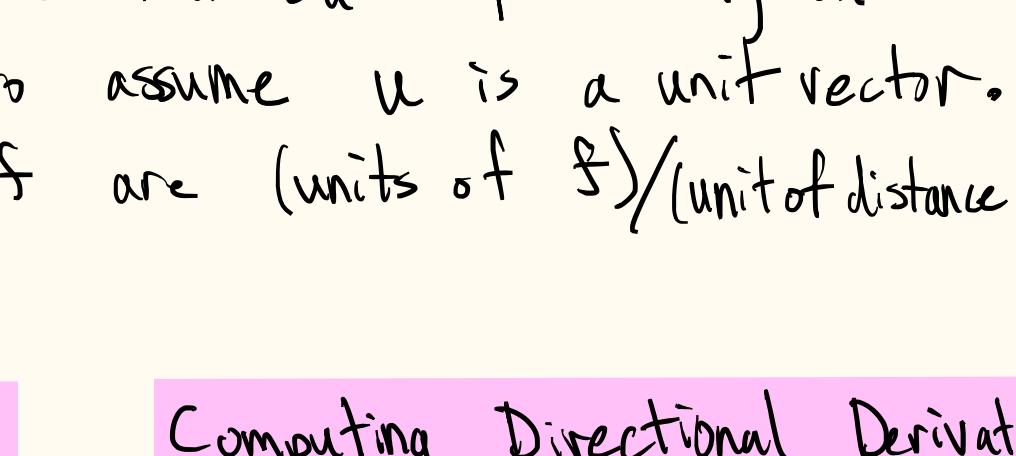
Suppose we travel a small distance h in the direction of $\mathbf{u}$.



Then we move from (x_0, y_0) to $(x_0 + hu_1, y_0 + hu_2)$.

The avg rate of change of f in the direction of $\mathbf{u}$ is:

$$\frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h} = \text{slope of red line}$$



By taking a limit $h \rightarrow 0$ we get the instantaneous rate of change of f in the $\mathbf{u}$ direction.

Definition 10.6.2

Let $f(x, y)$ be a function and $\mathbf{u} = \langle u_1, u_2 \rangle$ a unit vector. The **directional derivative** of f at (x_0, y_0) in the direction of $\mathbf{u}$ is the limit

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

whenever this limit exists.

In order to ensure that $D_{\mathbf{u}} f$ depends only on direction, it is important to assume $\mathbf{u}$ is a unit vector. The units of $D_{\mathbf{u}} f$ are (units of f)/(unit of distance).

Section 10.6.2

Computing Directional Derivative

We can obtain a formula for $D_{\mathbf{u}} f$ using the chain rule. Parametrize the line through (x_0, y_0) in the direction of $\mathbf{u}$:

$$\mathbf{r}(t) = \langle x_0 + u_1 t, y_0 + u_2 t \rangle$$

$$\text{Then } D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(x(0+h), y(0+h)) - f(x(0), y(0))}{h}$$

$$= \frac{d}{dt} f(x(t), y(t)) \Big|_{t=0}$$

$$\frac{dx}{dt} \Big|_t = \frac{dy}{dt} \Big|_t$$

$$= \frac{\partial f}{\partial x} \left(\left. \frac{dx}{dt} \right|_{t=0} \right) + \frac{\partial f}{\partial y} \left(\left. \frac{dy}{dt} \right|_{t=0} \right)$$

$$= f_x(x_0, y_0) u_1 + f_y(x_0, y_0) u_2$$

$$= \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \mathbf{u}.$$

$$\text{More compactly: } D_{\mathbf{u}} f = \langle f_x, f_y \rangle \cdot \mathbf{u}$$

Activity 10.6.2

- Complete Activity 10.6.2 and discuss w/ your group.
- Discuss as a class.

$$(a) \quad f_x = 3y - 2xy^3$$

$$f_y = 3x - 3x^2 y^2$$

$$(b) \quad D_{\mathbf{z}} f = \langle f_x, f_y \rangle \cdot \langle 1, 0 \rangle = f_x$$

$$D_{\mathbf{y}} f = \langle f_x, f_y \rangle \cdot \langle 0, 1 \rangle = f_y.$$

Partial derivatives are a special case of directional derivative.

$$(c) \quad D_{\mathbf{u}} f(1, -1) = \langle f_x(1, -1), f_y(1, -1) \rangle \cdot \left\langle \frac{2}{\sqrt{5}}, \frac{3}{\sqrt{5}} \right\rangle$$

$$= \langle -1, 0 \rangle = -\frac{2}{\sqrt{5}}$$

Section 10.6.3

The Gradient

Definition Let $f(x, y)$ be a function. The **gradient** of f at (x_0, y_0) is the vector

$$\nabla f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle.$$

You can also think of ∇f as a function $\mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$\nabla f = \langle f_x, f_y \rangle$$

The symbol ∇ is pronounced "del". w/ this notation, the directional derivative is

$$D_{\mathbf{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \mathbf{u}$$

or

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$$

where $\mathbf{u}$ is a unit vector.

Activity 10.6.3

- Complete Activity 10.6.3 and discuss w/ your group.
- Class discussion.

Activity 10.6.3. Let's consider the function f defined by $f(x, y) = x^2 - y^2$.

Some contours for this function are shown in Figure 10.6.3.

$$(a) \quad \nabla f = \langle 2x, -2y \rangle$$

$$(b) \quad \nabla f(2, 0) = \langle 4, 0 \rangle$$

$$\nabla f(0, 2) = \langle 0, -4 \rangle$$

$$\nabla f(2, 1) = \langle 4, -4 \rangle$$

$$\nabla f(2, -1) = \langle 4, 2 \rangle$$

$$\nabla f(3, 2) = \langle -6, -4 \rangle$$

$$\nabla f(-2, -4) = \langle -4, 8 \rangle$$

$$\nabla f(0, 0) = \langle 0, 0 \rangle$$

(c) It looks like $\nabla f(x_0, y_0)$ is $\perp$ to the level curve $f(x, y) = k$ where $k = f(x_0, y_0)$.

(d) From the picture, f is increasing in the direction of ∇f because the contours increase in elevation in that direction.

Section 10.6.4

Direction of the Gradient

The gradient $\nabla f(x_0, y_0)$ is a vector. From the activity we think it is perpendicular to the level curve of f at (x_0, y_0) .

Claim: $\nabla f(x_0, y_0)$ is $\perp$ to the level curve $f(x, y) = k$ where $k = f(x_0, y_0)$.

Parametrize the curve $f(x, y) = k$ via

$$\mathbf{r}(t) = \langle x(t), y(t) \rangle.$$

$$k = f(\mathbf{r}(t)) = f(x(t), y(t)).$$

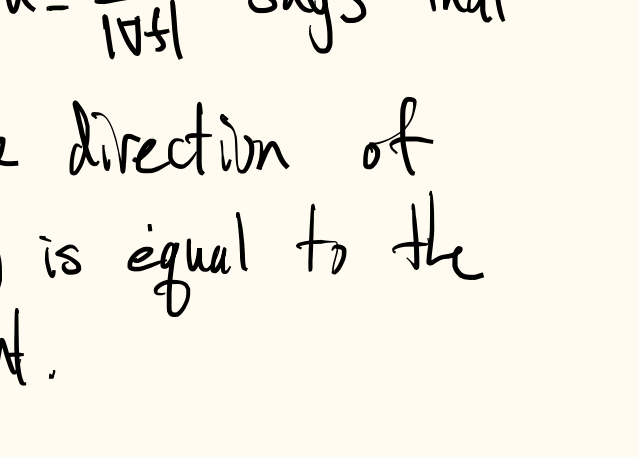
by definition of the level curve.

Differentiate both sides w.r.t. t .

$$0 = \frac{d}{dt} k = \frac{d}{dt} f(x(t), y(t))$$

$$\stackrel{\text{chain rule}}{=} \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt}$$

$$= \nabla f \cdot \mathbf{r}'(t).$$



The gradient $\nabla f(x_0, y_0)$ is $\perp$ to the level curve at the point (x_0, y_0) .

More generally, $\nabla f(x_0, y_0, z_0)$ is $\perp$ to the level surface through the point (x_0, y_0, z_0) defined by

$$f(x_0, y_0, z_0) = f(x, y, z).$$

We can use this fact to find normal vector for the tangent plane.

Tangent plane to a graph:

Let $f(x, y)$ be a function. We can use the gradient to find a normal vector for the tangent plane to the graph of $f(x, y)$.

Consider $g(x, y, z) = f(x, y) - z$. The level surface $0 = g(x, y, z)$ is just the graph of $f(x, y)$. By above, $\nabla g(x, y, z) = \langle f_x, f_y, -1 \rangle$ is normal to the graph of f .

So the tangent plane is

$$\langle f_x, f_y, -1 \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$



Activity 10.6.4

- Complete Activity 10.6.4 and discuss w/ your group.
- Class discussion.

$$(a) \quad D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$$

$$= |\nabla f| \cdot |\mathbf{u}| \cdot \cos(\theta)$$

$$= |\nabla f| \cos(\theta)$$

$$(b) \quad \cos(\theta) \text{ is maximal when } \theta = 0$$

and $|\nabla f|$ does not depend on θ

(c) When $\theta = 0$, $\mathbf{u}$ is parallel to $\nabla f(x_0, y_0)$. The directional derivative $D_{\mathbf{u}} f$ measures the instantaneous rate of change of f in the direction of $\mathbf{u}$. When $\mathbf{u}$ is parallel to ∇f , $D_{\mathbf{u}} f$ has the largest possible value. So the direction of greatest increase is in the direction of ∇f .

(d) The direction of steepest descent is $-\nabla f$ since formula in (a) is minimized when $\theta = 180^\circ$.

$$(e) \quad \frac{\nabla f}{|\nabla f|} \text{ and } -\frac{\nabla f}{|\nabla f|}, \quad \nabla f \neq \langle 0, 0 \rangle.$$

Section 10.6.5

Magnitude of the Gradient

The direction of steepest ascent is a unit vector in the direction of ∇f $\mathbf{u} = \frac{\nabla f}{|\nabla f|}$.

What is the rate of increase in the direction of $\mathbf{u}$?

$$D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u} = \nabla f \cdot \frac{\nabla f}{|\nabla f|} = \frac{|\nabla f|^2}{|\nabla f|} = |\nabla f|$$

The equation $D_{\mathbf{u}} f = |\nabla f|$ w/ $\mathbf{u} = \frac{\nabla f}{|\nabla f|}$ says that the rate of change of f in the direction of steepest ascent at a point (x_0, y_0) is equal to the length of the gradient at that point.